Second order corrections for the limits of normalized ruin times in the presence of heavy tails

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Abstract

In this paper we consider a compound Poisson risk model with regularly varying claim sizes. For this model in [4] an asymptotic formula for the finite time ruin probability is provided when the time is scaled by the mean excess function. In this paper we derive the rate of convergence for this finite time ruin probability when the claims have a finite second moment.

Keywords: Second order asymptotic; Regular variation; Finite time ruin probability; Poisson process; Risk Process; Transient behavior; M/G/1 queue; Storage process;

1 Introduction

In this paper, we consider the classical Cramér Lundberg risk process with (for convenience) constant premium inflow 1, claims $X_1, X_2, \ldots$ which are iid random variables with distribution F and arrive at the epochs of a Poisson process N_t with parameter λ and independent of the X_i. Denote with

$$S_t = \sum_{i=1}^{N_t} X_i - t$$

the claim surplus process at time t and with

$$\tau_u = \inf\{t | u - S_t < 0\}$$

the time of ruin with starting capital u. We are interested in the finite time ruin probability

$$\psi(u,t) = \mathbb{P}(\tau_u < t).$$

Denote with $\mu = \mathbb{E}[X]$ and

$$F_0(x) = \frac{1}{\mu}\int_0^x \overline{F}(x)\mathrm{d}x$$

*This work was partially supported by a grant from the Thiele Center at Aarhus University, by the the MIRACCLE-GICC project and the Chaire d'excellence "Generali - Actuariat responsable: gestion des risques naturels et changements climatiques."

the integrated tail distribution of F. We assume the usual net profit condition $\rho = \lambda\mu < 1$ ensuring that the ruin in infinite time does not occur w.p. 1. See for example [2].
In [4] (see also [2, Section X.4]) it is shown that if $\overline{F}_0$ is subexponential and there exists a non-degenerate random variable W and a function $e(u)$ such that

$$\lim_{u\to\infty} \frac{\overline{F}_0(u + xe(u))}{\overline{F}_0(u)} = \mathbb{P}(W > x), \tag{1.1}$$

then

$$\psi(u, xe(u)) \sim \frac{\rho}{1-\rho}\overline{F}_0(u)\mathbb{P}\left(\frac{W}{1-\rho} \leq x\right) \tag{1.2}$$

as $u \to \infty$ (see also [3], [15] and the discussion in [2, p. 318] for further work in this direction).
In this paper we want to give asymptotic expressions for the error in the approximation (1.2). Condition (1.1) (c.f. [12]) and results on second order asymptotic approximations for compound sums (cf. [1] for a recent survey) imply that we have to expect three different cases: $\overline{F}_0$ is regularly varying and has finite mean, $\overline{F}_0$ is regularly varying and has infinite mean, F_0 is in the maximum domain of attraction of the Gumbel distribution. In this paper we will only consider the first case, where W is regularly varying with finite mean (see further Assumption 1.1 below).
It should be noted that the our results also have some relevance for queueing and inventory theory. This is because of the relation between the Cramér-Lundberg model and a dual M/G/1 queue defined by the same arrival process and service times distributed as the X_i: $\psi(u,t) = \mathbb{P}(V_t > u)$ where V_t is the workload process in an initially empty queue (see [2, pp. 45–48]). This process is also frequently used as a storage process model.
We start the paper in Section 2 with a survey of recent result on second order subexponentiasl asymptotics. Section 3 then contain the statement of our main result. In addition we give the outline of the proof, which has many very technical steps (though often the crux is just careful Taylor expansions). This proof in turn is modeled after that of [4], where the simple and explicit ladder structure of the Cramér-Lundberg process plays a key role. We also give some discussion of the difficulties in extending to more general models such as Lévy processes or renewal models.
The proofs of the technical estimates omitted in Section 3 then occupy the rest of the paper. A longer version of the paper with some more detail given is available upon request from the authors.

2 Subexponential distributions and second order properties

In this paper we will assume that the distribution function F of X is regularly varying with index α, i.e.

$$\lim_{u\to\infty} \frac{\mathbb{P}(X > xu)}{\mathbb{P}(X > u)} = \lim_{u\to\infty} \frac{\overline{F}(xu)}{\overline{F}(X > u)} = x^{-\alpha}.$$

For more information about regularly varying we refer to [9]. Let $X_1, \ldots, X_n$ be iid copies of X denote with $S_n = \sum_{i=1}^n X_i$ and $M_n = \max_{i=1,\ldots,n} X_i$. The regularly varying

distributions are a subclass of the subexponential distributions defined through

$$\lim_{u\to\infty}\frac{\mathbb{P}(S_n>u)}{\mathbb{P}(X_1>u)}=\lim_{u\to\infty}\frac{\mathbb{P}(M_n>u)}{\mathbb{P}(X_1>u)}=n. \tag{2.1}$$

A basic result on second order asymptotics for subexponential distributions concerns the rate of convergence in (2.1).

If $\mathbb{E}[X]<\infty$ and F has a regularly varying density f, then it is shown in [19] that

$$\mathbb{P}(S_n>u)=n\overline{F}(u)+n(n-1)\mathbb{E}[X_1]\,f(u)+o\left(f(u)\right). \tag{2.2}$$

The regularly varying case with $\mathbb{E}[X]=\infty$ is treated in [18].
In [5] the result (2.2) is generalized to a wide class of subexponential distributions. Further it is pointed out in [1], that a Taylor expansion shows that (2.2) is equivalent to

$$\mathbb{P}(S_n>u)=n\overline{F}\left(u-(n-1)\mathbb{E}[X_1]\right)+o\left(f(u)\right)\,,$$

which has the natural interpretation that the sum is large if one component is large and the others behave in a normal way. One should note that in the cited references n can be a (light tailed) random variable. Hence by the Pollaczeck-Khinchine formula these results directly translate to second order results for the infinite time ruin probability.
Higher order expansions are provided in [6] and [7]; for a recent survey of this topic, see [1].
Extensions of these results are given in [11] where second order properties for the value-at-risk are provided. [10] considered the absolute ruin probability in a model where the insurance company can borrow money. In [17] dependent but tail independent regularly varying random variables are studied, and in [8] second order properties for the value-at-risk, when the risks are dependent according to an Archimedean copula, are provided.
Studies in the subexponential area often use the relation to extreme value theory, in our case the fact that condition (1.1) is equivalent to the condition that F_0 is in the maximum domain of attraction of the Fréchet extreme value distribution (see e.g. [12]). However, we will not use this connection.

3 Preliminaries and main theorem

To fix notation we present the idea of the proof of (1.2) with the notation and the method given in [2]. Therefore denote with

$$\tau_+(0)=0,\quad \tau_+(i)=\inf\{t>\tau_+(i-1)\,:\,S_t>S_{\tau_+(i-1)})\},\quad i\geq 1$$

the time of the i-th ladder step.
Further denote with $Y_i=S_{\tau_+(i)}-S_{\tau_+(i-1)}$ and $Z_i=S_{\tau_+(i-1)}-S_{\tau_+(i)-}$ the overshoot, resp. the capital before each ladder step. It is known that the (Y_i,Z_i) form a sequence of iid random vectors with joint distribution given by $\mathbb{P}(Y>y,Z>z)=\overline{F}_0(y+z)$. Denote with

$$K(u)=\inf\{n:\tau_+(n)<\infty,Y_1+\cdots+Y_n>u\}$$

the number of ladder steps until the time of ruin and with $\mathbb{P}^{(u,n)} = \mathbb{P}(\cdot|\tau(u) < \infty, K(u) = n)$

Denote with R_t a stochastic process independent of S_t and $R_t \overset{d}{=} -S_t$. Let $w(x) = \inf\{t : R_t = x\}$ the first time that the process R_t reaches level x. Under the measure $\mathbb{P}^{(u,n)}$ the distribution of $\tau(u)$ is the same as the one of $w(Z_1)+\cdots+w(Z_n)$ and $w(Z_1+\cdots+Z_n)$. Hence it follows that for $Z_1, \dots, Z_n|K(u) = n$ distributed according to $P^{(u,n)}$, the distribution of $\tau(u)$ is the same as the distribution of $w(Z_1 + \cdots + Z_{K(u)})$. So the method of proof for (1.2) is first to find the distribution of $Z_1, \dots, Z_n$ and then find the connection between $w(A)$ and A for some random variable A.

We will use the same ideas to prove our main results. We will work under the following Assumption which will be assumed to hold throughout the paper.

Assumption 3.1. *Let $X, X_1, X_2, \dots$ be a sequence of iid random variables with distribution function F having a regularly varying tail with index α, a regularly varying density f and Laplace transform $\hat{F}(s) = \int_0^\infty \mathrm{e}^{-sx} f(x)\mathrm{d}x$. Assume that $\mathbb{E}\left[X^2\right] < \infty$ and that there exists an $M > 0$ with $s\hat{F}(s) < M$ for $\mathrm{Re}(s) > 0$ and $|s| < 1$.*

It follows in particular that, taking $e(u) = u$, the r.v. W in (1.1) exists and has tail $\mathbb{P}(W > y) = (1 + y)^{-\alpha+1}$.

Theorem 3.1. *Let Assumption 3.1 be fulfilled and define $e(u) = u$. Then*

$$\begin{aligned}
\psi(u, xe(u)) = {} & \frac{\rho\overline{F}_0(u + x(1-\rho)e(u))}{(1-\rho)} + \frac{3\mathbb{E}\left[X^2\right]}{\mu}\frac{\rho^2\overline{F}(u + x(1-\rho)e(u))}{(1-\rho)^2} \\
& - \psi(u)\frac{\lambda\mathbb{E}\left[X_1^2\right]}{2e(u)(1-\rho)}\left(\frac{(\alpha-1)}{(1 + x(1-\rho))^{\alpha}} - \frac{\alpha(\alpha-1)x(1-\rho)}{(1 + x(1-\rho))^{\alpha+1}}\right) \\
& + o\left(\overline{F}(u)\right).
\end{aligned}$$

Remark 3.1. Using $\mathbb{P}(W > y) = (1 + y)^{-\alpha+1}$ and simple calculus, it is easy to see that the r.h.s. of (1.2) and the first term in the expansion of $\psi(u, xe(u))$ in Theorem 3.1 are both of order $c_1L(u)/u^{\alpha-1}$, with $L(u)$ the slowly varying function common for f, F, F_0 and $c_1 = \rho/\left[(1-\rho)\mu(\alpha-1)\right]$. The two next terms are, up to constants, both of order $L(u)/u^{\alpha}$.

Proof. We give the outline, with some lengthy and technical details being given later as Lemmas 5.1–5.8 and 6.1–6.3. From [2] we get that for $S_n = Y_1 + \cdots + Y_n$

$$\mathbb{P}(K(u) = n) = \frac{\rho^n}{\psi(u)}\mathbb{P}(S_n > u, S_{n-1} \le u).$$

From Lemma 6.1 we get that

$$\begin{aligned}
& \mathbb{P}^{(u,n)}(Z_1 + \cdots + Z_n > xe(u))\mathbb{P}(K(u) = n) \\
& = \frac{\rho^n\overline{F}_0(u + xe(u))}{\psi(u)} + \frac{3\mathbb{E}\left[(n-1)X^2\right]}{\mu}\frac{\rho^n\overline{F}(u + xe(u))}{\psi(u)} + o\left(\frac{\overline{F}(u)}{\psi(u)}\right).
\end{aligned}$$

Summing over n we get that

$$\begin{aligned}
& \mathbb{P}^{(u)}(Z_1 + \cdots + Z_{K(u)} > xe(u)) \\
& = \frac{\rho\overline{F}_0(u + xe(u))}{(1-\rho)\psi(u)} + \frac{3\mathbb{E}\left[X^2\right]}{\mu}\frac{\rho^2\overline{F}(u + xe(u))}{(1-\rho)^2\psi(u)} + o\left(\frac{\overline{F}(u)}{\psi(u)}\right).
\end{aligned}$$

From Lemmas 6.2 and 6.3 we get that

$$W_u = \frac{Z_1 + \cdots + Z_n}{x(1-\rho)e(u)}$$

fulfills the conditions of Lemma 5.1 and hence

$$\begin{aligned}\frac{\psi(u, xe(u))}{\psi(u)} &= \frac{\rho\overline{F}_0(u + x(1-\rho)e(u))}{(1-\rho)\psi(u)} + \frac{3}{\mu}\frac{\rho^2\overline{F}(u + x(1-\rho)e(u))}{(1-\rho)^2\psi(u)} \\ &\quad - \frac{\lambda\mathbb{E}\left[X_1^2\right]}{2e(u)(1-\rho)}\left(\frac{1}{x(1-\rho)}g_\infty(1) + \frac{1}{x(1-\rho)}g'_\infty(1)\right) \\ &\quad + o\left(\frac{1}{e(u)}\right) + o\left(\frac{\overline{F}(u)}{\psi(u)}\right)\end{aligned}$$

□

Remark 3.2. The proof of Theorem 3.1 relies on two observations. First, we used that the distribution of the sum of the surpluses before each ladder step has a known distribution, which is related to the distribution that a random sum exceeds a given threshold and hence we can use methods developed for random sums to get second order properties. The second fact that we used is that we know the connection between the time of ruin and the sum of the surpluses. This connection allows to involve the central limit theorem for compound Poisson sums (cf. Section 4) and hence higher order asymptotics can be found. These two properties of compound Poisson processes are not straightforward to generalize to more general risk models like renewal models since they heavily rely on the fact that the considered risk process is Markovian. Similarly, the extension to general Lévy processes meets the difficulty that the ladder structure here is more complicated.
Another interesting extension is to consider the case where F has finite mean but infinite variance. The difficulty here is that the CLT for Poisson sums has to be replaced with some sort of stable limit.

4 Some notation

The notation of this section will be used in the rest of the paper without further mentioning. Recall from Asssumption 3.1 that $\hat{F}(s) = \mathbb{E}\left[\mathrm{e}^{-sX}\right]$ is the Laplace transform of the claim size distribution F and let $\kappa(s) = s + \lambda(\hat{F}(s) - 1)$. Then we have

$$\mathbb{E}\left[\mathrm{e}^{-sS_t}\right] = \mathrm{e}^{t\kappa(s)}.$$

We get from [2, Lemma XI.3.1]

$$\mathbb{E}\left[\mathrm{e}^{-sw(z)}\right] = \mathrm{e}^{-\kappa^{-1}(s)z}.$$

For a function $g(x)$ we denote with $\hat{L}_g(s) = \int_0^\infty \mathrm{e}^{-sx} g(x)\mathrm{d}x$ the Laplace transform. Note that

$$\hat{L}_F(s) = \frac{1}{s}\hat{F}(s).$$

To study the distribution of $w(z)$, note that we can write

$$w(z) = z + \sum_{i=1}^{N(z)} E_i \,,$$

where the E_i are iid having the distribution of $E = w(X)$ (the E_i represent the excursions of R_t away from its running maximum). Also, as a sample path inspection immediately shows, E has the busy period distribution in the usual dual M/G/1 queue (see [2, pp. 45–48]). Since the Laplace transform is $\hat{F}_E(s) = \hat{F}(\kappa^{-1}(s))$, it follows that

$$\begin{aligned}
\mathbb{E}\left[E\right] &= \mathbb{E}\left[X\right]/(1-\lambda\mathbb{E}\left[X\right]) = \mathbb{E}\left[X\right]/(1-\rho) \\
\mathbb{E}\left[E^2\right] &= \frac{\mathbb{E}\left[X^2\right]}{(1-\rho)^2}\left(1+\frac{\lambda\mathbb{E}\left[X\right]}{1-\rho}\right) = \frac{\mathbb{E}\left[X^2\right]}{(1-\rho)^3}.
\end{aligned}$$

Write $h(z) = w(z) - z(\lambda\mathbb{E}\left[E\right]+1) = w(z) - z/(1-\rho)$ and $U(z) = h(z)/\sqrt{z}$. By the central limit theorem, $U(z) \to N(0, \lambda\mathbb{E}\left[E^2\right])$. Essentially, the claim size density f and the density of E have the same degree of smoothness. Since we don't want to postulate smoothness conditions on f, we will use smoothing with a normal random variable. Therefore denote with N_u a normal random variable with mean zero and variance $\sigma_u^2 = e(u)^{-4}$.

In the proofs of this paper we will often rely on Taylor approximations with remainder terms. Therefore we will need to evaluate a function on an interim value which we will denote with ξ_Θ where Θ stands for the parameters on which ξ dependents. With a little abuse of notation we will also use this notation when we use Taylor expansions for a complex function (in this case one would have for the real and the imaginary part a different ξ) and when the derivative is not continuous.

5 The connection between $w(W)$ and W

Lemma 5.1. *Let W_u be a family of random variables with distribution function $G_u(w)$ with $\lim_{u\to\infty} \overline{G}_u(w) = \overline{G}_\infty(w) = (1 + x(1-\rho)w)^{-\alpha+1}$. Further assume that W_u has a density $g_u(x)$ that is continuously differentiable and $\lim_{u\to\infty} g_u(w) = g_\infty(w)$ as well as $\lim_{u\to\infty} g_u'(w) = g_\infty'(w)$. Then*

$$\begin{aligned}
&\mathbb{P}(w((1-\rho)xe(u)W_u) > xe(u)) = \mathbb{P}(W_u > 1) \\
&\quad - \frac{\lambda\mathbb{E}\left[X_1^2\right]}{2e(u)(1-\rho)}\left(\frac{1}{x(1-\rho)}g_\infty(1) + \frac{1}{x(1-\rho)}g_\infty'(1)\right) + o\left(\frac{1}{e(u)}\right).
\end{aligned}$$

Proof. First consider $W_u > 1/(1-\epsilon)$. We get from Lemma A.5 that there exists an $\delta > 0$

with

$$
\begin{aligned}
&\mathbb{P}(W_u > 1/(1-\epsilon)) - \mathbb{P}(w((1-\rho)xe(u)W_u) > xe(u), W_u > 1/(1-\epsilon)) \\
&= \int_{1/(1-\epsilon)}^{\infty} \mathbb{P}\left(\frac{h((1-\rho)xe(u)w)}{(1-\rho)xe(u)w} \le \frac{1/w-1}{1-\rho}\right) \mathrm{d}G_u(w) \\
&\le \int_{1/(1-\epsilon)}^{\infty} \mathbb{P}\left(\frac{h((1-\rho)xe(u)w)}{(1-\rho)xe(u)w} \le -\frac{\epsilon}{1-\rho}\right) \mathrm{d}G_u(w) \\
&\le \int_{1/(1-\epsilon)}^{\infty} \mathrm{e}^{-\delta e(u)w} \mathrm{d}G_u(w) \\
&= o(e(u)^{-1}).
\end{aligned} \tag{5.1}
$$

For $W_u < 1/(1+\epsilon)$ we get by [16]

$$
\begin{aligned}
&\mathbb{P}(w((1-\rho)xe(u)W_u) > xe(u), W_u < 1/(1+\epsilon)) \\
&= \int_0^{1/(1+\epsilon)} \mathbb{P}\left(\frac{h((1-\rho)xe(u)w)}{(1-\rho)xe(u)w} > \frac{1/w-1}{1-\rho}\right) \mathrm{d}G_u(w) \\
&\sim \int_0^{1/(1+\epsilon)} \lambda(1-\rho)xe(u)w\mathbb{P}\left(E > (1-w)xe(u)\right) \mathrm{d}G_u(w) \\
&\le \lambda(1-\rho)xe(u)\mathbb{P}\left(E > \frac{\epsilon}{1+\epsilon}xe(u)\right) \int_0^{1/(1+\epsilon)} w\mathrm{d}G_u(w) \\
&= o(e(u)^{-1}).
\end{aligned} \tag{5.2}
$$

Finally we have to consider the case $1/(1+\epsilon) < W_u < 1/(1-\epsilon)$.

$$
\begin{aligned}
&\mathbb{P}(w((1-\rho)xe(u)W_u) > xe(u), 1/(1+\epsilon) < W_u < 1/(1-\epsilon)) \\
&= \int_{1/(1+\epsilon)}^{1/(1-\epsilon)} \mathbb{P}\left(\frac{h((1-\rho)xe(u)w)}{(1-\rho)xe(u)w} > \frac{1/w-1}{1-\rho}\right) \mathrm{d}G_u(w) \\
&= \mathbb{P}(1 \le W_u < 1/(1-\epsilon)) \\
&\quad + \int_{1/(1+\epsilon)}^{1} \mathbb{P}\left(\frac{h((1-\rho)xe(u)w)}{(1-\rho)xe(u)w} > \frac{1/w-1}{1-\rho}\right) \mathrm{d}G_u(w) \qquad (5.3) \\
&\quad - \int_{1}^{1/(1-\epsilon)} \mathbb{P}\left(\frac{h((1-\rho)xe(u)w)}{(1-\rho)xe(u)w} \le \frac{1/w-1}{1-\rho}\right) \mathrm{d}G_u(w). \qquad (5.4)
\end{aligned}
$$

We start with (5.3). Denote with

$$
x(w,u) = \frac{x(1-\rho)}{1+(1-\rho)\frac{w}{\sqrt{e(u)}}}. \tag{5.5}
$$

For N_u normally distributed with mean 0 and variance $\sigma_u^2 = (xe(u))^{-4}$ we get from Lemma

A.2

$$
\begin{aligned}
&\int_{1/(1+\epsilon)}^{1} \mathbb{P}\left(\frac{h((1-\rho)xe(u)w)}{(1-\rho)xe(u)w} > \frac{1/w-1}{1-\rho}\right) \mathrm{d}G_u(w) \\
&= \frac{1-\rho}{\sqrt{e(u)}} \int_0^{\sqrt{e(u)}\epsilon/(1-\rho)} \mathbb{P}\left(\frac{h\left(x(w,u)e(u)\right)}{\sqrt{x(w,u)e(u)}} > w\sqrt{x(w,u)}\right) \frac{g_u\left(\frac{1}{1+(1-\rho)\frac{w}{\sqrt{e(u)}}}\right)}{\left(1+(1-\rho)\frac{w}{\sqrt{e(u)}}\right)^2} \mathrm{d}w \\
&= \frac{1-\rho}{\sqrt{e(u)}} \int_0^{\sqrt{e(u)}\epsilon/(1-\rho)} \mathbb{P}\left(N_u + \frac{h\left(x(w,u)e(u)\right)}{\sqrt{x(w,u)e(u)}} > w\sqrt{x(w,u)}\right) \\
&\quad \times \frac{g_u\left(\frac{1}{1+(1-\rho)\frac{w}{\sqrt{e(u)}}}\right)}{\left(1+(1-\rho)\frac{w}{\sqrt{e(u)}}\right)^2} \mathrm{d}w + o\left(\frac{1}{e(u)}\right) \\
&= \frac{1-\rho}{\sqrt{e(u)}} \int_0^{\sqrt{e(u)}\epsilon/(1-\rho)} \mathbb{P}\left(N_u + \frac{h\left(x(w,u)e(u)\right)}{\sqrt{x(w,u)e(u)}} > w\sqrt{x(w,u)}\right) g_u\left(1\right) \mathrm{d}w \qquad (5.6) \\
&\quad - \frac{(1-\rho)^2}{e(u)} \int_0^{\sqrt{e(u)}\epsilon/(1-\rho)} w\mathbb{P}\left(N_u + \frac{h\left(x(w,u)e(u)\right)}{\sqrt{x(w,u)e(u)}} > w\sqrt{x(w,u)}\right) \qquad (5.7) \\
&\quad \times \left(\frac{g_u'\left(\frac{1}{1+\xi_{u,w}}\right)}{(1+\xi_{u,w})^4} + 2\frac{g_u\left(\frac{1}{1+\xi_{u,w}}\right)}{(1+\xi_{u,w})^3}\right) \mathrm{d}w + o\left(\frac{1}{e(u)}\right).
\end{aligned}
$$

We have to evaluate the integrals in (5.6) and (5.7) so we split the integrals into $\int_0^M$ and $\int_M^{\sqrt{e(u)}\epsilon/(1-\rho)}$. By Lemma 5.2 we get that

$$
\begin{aligned}
&\frac{1-\rho}{\sqrt{e(u)}} \int_0^M \mathbb{P}\left(N_u + \frac{h\left(x(w,u)e(u)\right)}{\sqrt{x(w,u)e(u)}} > w\sqrt{x(w,u)}\right) g_u\left(1\right) \mathrm{d}w \\
&= \frac{1-\rho}{\sqrt{e(u)}} \int_0^M \mathbb{P}\left(N_u + \frac{h\left(x(0,u)e(u)\right)}{\sqrt{x(0,u)e(u)}} > w\sqrt{x(0,u)}\right) g_u\left(1\right) \mathrm{d}w \\
&\quad + \frac{(1-\rho)^2}{e(u)} \int_0^M w^2 \sqrt{x(1-\rho)} f_{0,\infty}(w\sqrt{x(1-\rho)}) g_u\left(1\right) \mathrm{d}w \\
&\quad + \frac{(1-\rho)^2}{2e(u)} \lambda \mathbb{E}\left[E^2\right] \int_0^M w f_{w,\infty}'\left(w\sqrt{x(1-\rho)}\right) g_u\left(1\right) \mathrm{d}w \\
&\quad + o(1/e(u)).
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\lim_{M\to\infty} \int_0^M w^2 \sqrt{x(1-\rho)} f_{w,\infty}(w\sqrt{x(1-\rho)}) g_\infty\left(1\right) \mathrm{d}w = \frac{g_\infty(1)\lambda\mathbb{E}\left[E^2\right]}{2x(1-\rho)} \\
&\lim_{M\to\infty} \int_0^M w f_{w,\infty}'\left(w\sqrt{x(1-\rho}\right) g_\infty\left(1\right) \mathrm{d}w = -\frac{g_\infty(1)}{2x(1-\rho)}.
\end{aligned}
$$

$$
\begin{aligned}
\lim_{M\to\infty}\lim_{u\to\infty}\int_0^M & w\mathbb{P}\left(N_u+\frac{h\left(x(w,u)e(u)\right)}{\sqrt{x(w,u)e(u)}}>w\sqrt{x(w,u)}\right)\\
&\times\left(\frac{g_u'(1+\xi_{u,w})}{(1+\xi_{u,w})^2}+2\frac{g_u(1+\xi_{u,w})}{(1+\xi_{u,w})^3}\right)\mathrm{d}w=\frac{\lambda\mathbb{E}\left[E^2\right]}{4x(1-\rho)}\left(g_\infty'(1)+2g_\infty(1)\right).
\end{aligned}
$$

For the integral $\int_M^{\sqrt{e(u)}\epsilon/(1-\rho)}$ we get from Lemmas 5.3 and 5.4 that there exist a function $R(M,u)\lesssim\frac{C_M}{e(u)}$ and $C_M\to 0$ as $u\to\infty$ such that the sum of (5.6) and (5.7) is the same as

$$
\frac{1-\rho}{\sqrt{e(u)}}\int_M^{\sqrt{e(u)}\epsilon/(1-\rho)}\mathbb{P}\left(N_u+\frac{h\left(x(0,u)e(u)\right)}{\sqrt{x(0,u)e(u)}}>w\sqrt{x(0,u)}\right)g_\infty\left(1\right)\mathrm{d}w+R(M,u).
$$

With Lemmas 5.5, 5.6 and 5.7, we can get analogously the asymptotic of (5.4), so that we are left with the integrals

$$
\begin{aligned}
&\int_0^{\sqrt{e(u)}\epsilon/(1-\rho)}\mathbb{P}\left(N_u+\frac{h\left(x(0,u)e(u)\right)}{\sqrt{x(0,u)e(u)}}>w\sqrt{x(0,u)}\right)\mathrm{d}w\\
&-\int_0^{\sqrt{e(u)}\epsilon/(1-\rho)}\mathbb{P}\left(N_u+\frac{h\left(x(0,u)e(u)\right)}{\sqrt{x(0,u)e(u)}}\leq -w\sqrt{x(0,u)}\right)\mathrm{d}w.
\end{aligned}
$$

From Lemma 5.8 we get that the last equation is asymptotically negligibility and hence the Lemma follows.

□

Lemma 5.2. *Under Assumption 3.1 we get uniformly for $w<M$ that*

$$
\begin{aligned}
\mathbb{P}\left(N_u+\frac{h\left(x(w,u)e(u)\right)}{\sqrt{x(w,u)e(u)}}>w\sqrt{x(w,u)}\right)&=\mathbb{P}\left(N_u+\frac{h\left(x(0,u)e(u)\right)}{\sqrt{x(0,u)e(u)}}>w\sqrt{x(0,u)}\right)\\
&+w^2\sqrt{x(1-\rho)}\frac{1-\rho}{\sqrt{e(u)}}f_{0,\infty}(w\sqrt{x(1-\rho)})\\
&+\frac{(1-\rho)w}{2\sqrt{e(u)}}\lambda\mathbb{E}\left[E^2\right]f_{w,\infty}'\left(w\sqrt{x(1-\rho}\right)\\
&+o(1/\sqrt{e(u)}).
\end{aligned}
$$

where $x(w,u)$ is defined by (5.5) *and $f_{w,\infty}$ is the density of a normal distribution with mean* 0 *and variance $\lambda\mathbb{E}\left[E^2\right]$.*

Proof. Denote with $f_{w,u}(x)$ the density of

$$
Z_u=N_u+\frac{h\left(x(w,u)e(u)\right)}{\sqrt{x(w,u)e(u)}}.
$$

and with $\hat{f}_{w,u}(x)$ the density of

$$
\hat{Z}_u=\hat{N}_u+\frac{h\left((x(0,u)-x(w,u))e(u)\right)}{\sqrt{(x(0,u)-x(w,u))e(u)}}.
$$

where $\hat{N}_u$ is an independent copy of $N(u)$. Z_u and $\hat{Z}_u$ are independent. Since $x(w,u)$ is monotonically decreasing in w, we get that

$$\begin{aligned}
&\mathbb{P}\left(N_u + \frac{h\left(x(0,u)e(u)\right)}{\sqrt{x(0,u)e(u)}} > w\frac{x(w,u)}{\sqrt{x(0,u)}}\right) \\
&= \mathbb{P}\left(N_u + \frac{h\left(x(w,u)e(u)\right) + h\left((x(0,u)-x(w,u))e(u)\right)}{\sqrt{x(0,u)e(u)}} > w\frac{x(w,u)}{\sqrt{x(0,u)}}\right) \\
&= \mathbb{P}\left(Z_u + \sqrt{\frac{(1-\rho)w}{\sqrt{e(u)}}}\hat{Z}_u > w\sqrt{x(w,u)}\right) \\
&= \mathbb{P}\left(Z_u > w\sqrt{x(w,u)}\right) + \sqrt{\frac{(1-\rho)w}{\sqrt{e(u)}}}\mathbb{E}\left[\hat{Z}_u\right] f_{w,u}(w\sqrt{x(w,u)}) \\
&\quad - \frac{(1-\rho)w}{2\sqrt{e(u)}}\mathbb{E}\left[\hat{Z}_u^2 f'_{w,u}(w\sqrt{x(w,u)} + \xi_{w,u})\right] \\
&= \mathbb{P}\left(Z_u > w\sqrt{x(w,u)}\right) - \frac{(1-\rho)w}{2\sqrt{e(u)}}\mathbb{E}\left[\hat{Z}_u^2\right] f'_{w,\infty}\left(w\sqrt{x(1-\rho}\right) + o\left(\frac{1}{\sqrt{e(u)}}\right),
\end{aligned}$$

here the last equality follows by bounded convergence. Finally note that

$$\begin{aligned}
&\mathbb{P}\left(N_u + \frac{h\left(x(0,u)e(u)\right)}{\sqrt{x(0,u)e(u)}} > w\frac{x(w,u)}{\sqrt{x(0,u)}}\right) \\
&= \mathbb{P}\left(N_u + \frac{h\left(x(0,u)e(u)\right)}{\sqrt{x(0,u)e(u)}} > w\sqrt{x(0,u)}\right) \\
&\quad + \left(w\frac{x(0,u)-x(w,u)}{\sqrt{x(0,u)}}\right) f_{0,u}(w\sqrt{x(0,u)} + \xi_{u,w}) \\
&= \mathbb{P}\left(N_u + \frac{h\left(x(0,u)e(u)\right)}{\sqrt{x(0,u)e(u)}} > w\sqrt{x(0,u)}\right) \\
&\quad + w^2\sqrt{x(1-\rho)}\frac{1-\rho}{\sqrt{e(u)}} f_{0,\infty}(w\sqrt{x(0,u)}) + o\left(\frac{1}{\sqrt{e(u)}}\right).
\end{aligned}$$

$\square$

Lemma 5.3. *Under Assumption 3.1 we get that for every $c > 0$ uniformly for $M \leq w < c\sqrt{e(u)}$ that*

$$\begin{aligned}
&\left|\mathbb{P}\left(N_u + \frac{h\left(x(w,u)e(u)\right)}{\sqrt{x(w,u)e(u)}} > w\sqrt{x(w,u)}\right) - \mathbb{P}\left(N_u + \frac{h\left(x(0,u)e(u)\right)}{\sqrt{x(0,u)e(u)}} > w\frac{x(w,u)}{\sqrt{x(0,u)}}\right)\right| \\
&\leq \frac{C_1}{w^2\sqrt{e(u)}} + C_2 w\sqrt{e(u)}\mathbb{P}\left(E > \frac{\epsilon_1}{2}w\sqrt{e(u)}\sqrt{(1-\rho)x(w,u)}\right) + o\left(\frac{1}{e(u)}\right)
\end{aligned}$$

where $x(w,u)$ is defined by (5.5).

Proof. We use the notation of the proof of Lemma 5.2.
Choose an $0 < \epsilon_1 < 1$. Since $x(w,u)$ is monotonically decreasing in w, we get that

$$
\begin{aligned}
&\mathbb{P}\left(N_u + \frac{h\left(x(0,u)e(u)\right)}{\sqrt{x(0,u)e(u)}} > w\frac{x(w,u)}{\sqrt{x(0,u)}}\right)\\
&= \mathbb{P}\left(Z_u + \sqrt{\frac{(1-\rho)w}{\sqrt{e(u)}}}\hat{Z}_u > w\sqrt{x(w,u)}, |\hat{Z}_u| \le \epsilon_1\sqrt{w\sqrt{e(u)}}\right)\\
&\quad + \mathbb{P}\left(Z_u + \sqrt{\frac{(1-\rho)w}{\sqrt{e(u)}}}\hat{Z}_u > w\sqrt{x(w,u)}, |\hat{Z}_u| > \epsilon_1\sqrt{w\sqrt{e(u)}}\right).
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\mathbb{P}\left(Z_u + \sqrt{\frac{(1-\rho)w}{\sqrt{e(u)}}}\hat{Z}_u > w\sqrt{x(w,u)}, |\hat{Z}_u| \le \epsilon_1\sqrt{w\sqrt{e(u)}}\right)\\
&= \mathbb{P}\left(Z_u > w\sqrt{x(w,u)}, |\hat{Z}_u| \le \epsilon_1\sqrt{w\sqrt{e(u)}}\right)\\
&+ \sqrt{\frac{(1-\rho)w}{\sqrt{e(u)}}}\mathbb{E}\left[\hat{Z}_u 1_{\left\{|\hat{Z}_u| \le \epsilon_1\sqrt{w\sqrt{e(u)}}\right\}}\right] f_{w,u}(w\sqrt{x(w,u)})\\
&\quad + \frac{(1-\rho)w}{\sqrt{e(u)}}\mathbb{E}\left[\hat{Z}_u^2 f'_{w,u}(w\sqrt{x(w,u)} + \xi_{w,u}) 1_{\left\{|\hat{Z}_u| \le \epsilon_1\sqrt{w\sqrt{e(u)}}\right\}}\right].
\end{aligned}
$$

Since $\mathbb{E}\left[\hat{Z}_u\right] = 0$ we get that

$$
\begin{aligned}
\left|\mathbb{E}\left[\hat{Z}_u 1_{\left\{|\hat{Z}_u| \le \epsilon_1\sqrt{w\sqrt{e(u)}}\right\}}\right]\right| &= \left|\mathbb{E}\left[\hat{Z}_u 1_{\left\{|\hat{Z}_u| > \epsilon_1\sqrt{w\sqrt{e(u)}}\right\}}\right]\right|\\
&\le \frac{1}{\epsilon_1\sqrt{w\sqrt{e(u)}}}\left|\mathbb{E}\left[\hat{Z}_u^2 1_{\left\{|\hat{Z}_u| > \epsilon_1\sqrt{w\sqrt{e(u)}}\right\}}\right]\right|.
\end{aligned}
$$

By Lemma A.3 $x^2 f_{w,u}(x)$ is bounded and hence for some $c_1 > 0$

$$
\sqrt{\frac{(1-\rho)w}{\sqrt{e(u)}}}\mathbb{E}\left[\hat{Z}_u 1_{\left\{|\hat{Z}_u| \le \epsilon_1\sqrt{w\sqrt{e(u)}}\right\}}\right] f_{w,u}(w\sqrt{x(w,u)})) \le c_1\mathbb{E}\left[\hat{Z}_u^2\right]\frac{1}{w^2\sqrt{e(u)}}.
$$

Denote with

$$
a = \left(\frac{(1-\rho)x}{1+c(1-\rho)} - \epsilon_1\sqrt{1-\rho}\right) \quad \text{and} \quad b = \left((1-\rho)x + \epsilon_1\sqrt{1-\rho}\right).
$$

We will assume that ϵ_1 is chosen such that $a > 0$. From

$$
\mathbb{E}\left[\hat{Z}_u^2 f'_{w,u}(w\sqrt{x(w,u)}) + \xi_{w,u}) 1_{\left\{|\hat{Z}_u| \le \epsilon_1\sqrt{w\sqrt{e(u)}}\right\}}\right] \le \mathbb{E}\left[\hat{Z}_u^2\right] \sup_{aw<x<bw} f'_{w,u}(x)
$$

and Lemma A.3 we get that $\sup_{aw<x<bw} f'_{w,u}(x) \le c_2/w^3$ and for some $c_3 > 0$

$$\frac{(1-\rho)w}{\sqrt{e(u)}}\mathbb{E}\left[\hat{Z}_u^2 f'_{w,u}(w\sqrt{x(w,u)}) + \xi_{w,u})1_{\left\{|\hat{Z}_u| \le \epsilon_1\sqrt{w\sqrt{e(u)}}\right\}}\right] \le c_3\mathbb{E}\left[\hat{Z}_u^2\right]\frac{1}{w^2\sqrt{e(u)}}.$$

Further we have with Lemma A.5 and $\mathbb{P}(|X+Y| > u) \le \mathbb{P}(|X| > u/2) + \mathbb{P}(|Y| > u/2)$ that for a standard normal distributed random variable $\mathcal{N}$

$$\begin{aligned}
&\mathbb{P}\left(Z_u + \sqrt{\frac{(1-\rho)w}{\sqrt{e(u)}}}\hat{Z}_u > w\sqrt{x(w,u)}, |\hat{Z}_u| > \epsilon_1\sqrt{w\sqrt{e(u)}}\right) \\
&\le \mathbb{P}\left(|\hat{Z}_u| > \epsilon_1\sqrt{w\sqrt{e(u)}}\right) \\
&\le C_2 w(1-\rho)x(w,u)\sqrt{e(u)}\sqrt{x(w,u)}\mathbb{P}\left(E > \frac{\epsilon_1}{2}w\sqrt{e(u)}\sqrt{(1-\rho)x(w,u)}\right) \\
&\quad + e^{-\delta\frac{\epsilon_1}{4}w\sqrt{e(u)}\sqrt{(1-\rho)x(w,u)}} + \mathbb{P}\left(|\mathcal{N}| > \frac{\epsilon_1}{2}\sqrt{w\sqrt{e(u)}}\right).
\end{aligned}$$

□

Lemma 5.4. *Under Assumption 3.1 we get that*

$$\begin{aligned}
&\int_M^{c\sqrt{e(u)}} \mathbb{P}\left(N_u + \frac{h\left(x(0,u)e(u)\right)}{\sqrt{x(0,u)e(u)}} > w\frac{x(w,u)}{\sqrt{x(0,u)}}\right)\mathrm{d}w \\
&= \int_M^{c\sqrt{e(u)}} \mathbb{P}\left(N_u + \frac{h\left(x(0,u)e(u)\right)}{\sqrt{x(0,u)e(u)}} > w\sqrt{x(0,u)}\right)\mathrm{d}w + R(u,M),
\end{aligned}$$

where

$$R(u,M) \lesssim \frac{C_M}{\sqrt{e(u)}}$$

and $C_M \to 0$ as $M \to \infty$.

Proof. By substitution we get that

$$\int_M^{c\sqrt{e(u)}} \mathbb{P}\left(N_u + \frac{h\left(x(0,u)e(u)\right)}{\sqrt{x(0,u)e(u)}} > w\frac{x(w,u)}{\sqrt{x(0,u)}}\right)\mathrm{d}w$$

$$\int_{\frac{M}{1+M\frac{1-\rho}{\sqrt{e(u)}}}}^{\frac{c\sqrt{e(u)}}{1+c(1-\rho)}} \frac{1}{1-w\frac{1-\rho}{\sqrt{e(u)}}}\mathbb{P}\left(N_u + \frac{h\left(x(0,u)e(u)\right)}{\sqrt{x(0,u)e(u)}} > w\sqrt{x(0,u)}\right)\mathrm{d}w \tag{5.8}$$

$$+\frac{1-\rho}{\sqrt{e(u)}}\int_{\frac{M}{1+M\frac{1-\rho}{\sqrt{e(u)}}}}^{\frac{c\sqrt{e(u)}}{1+c(1-\rho)}} \frac{w}{\left(1-w\frac{1-\rho}{\sqrt{e(u)}}\right)^2}\mathbb{P}\left(N_u + \frac{h\left(x(0,u)e(u)\right)}{\sqrt{x(0,u)e(u)}} > w\sqrt{x(0,u)}\right)\mathrm{d}w \tag{5.9}$$

(5.9) can be bounded by

$$\frac{1-\rho}{\sqrt{e(u)}}(1+c(1-\rho))^2\int_{\frac{M}{1+M\frac{1-\rho}{\sqrt{e(u)}}}}^{\infty} w\mathbb{P}\left(N_u + \frac{h\left(x(0,u)e(u)\right)}{\sqrt{x(0,u)e(u)}} > w\sqrt{x(0,u)}\right)\mathrm{d}w \sim \frac{c_M}{\sqrt{e(u)}}.$$

where $c_M \to 0$ as $M \to \infty$. For (5.8) we have

$$
\begin{aligned}
&\int_{\frac{M}{1+M\frac{1-\rho}{\sqrt{e(u)}}}}^{\frac{c\sqrt{e(u)}}{1+c(1-\rho)}} \frac{1}{1-w\frac{1-\rho}{\sqrt{e(u)}}} \mathbb{P}\left(N_u + \frac{h\left(x(0,u)e(u)\right)}{\sqrt{x(0,u)e(u)}} > w\sqrt{x(0,u)}\right) \mathrm{d}w \\
&= \int_{\frac{M}{1+M\frac{1-\rho}{\sqrt{e(u)}}}}^{\frac{c\sqrt{e(u)}}{1+c(1-\rho)}} \frac{1-\rho}{\sqrt{e(u)}} \frac{w}{1-w\frac{1-\rho}{\sqrt{e(u)}}} \mathbb{P}\left(N_u + \frac{h\left(x(0,u)e(u)\right)}{\sqrt{x(0,u)e(u)}} > w\sqrt{x(0,u)}\right) \mathrm{d}w \qquad (5.10) \\
&\quad + \int_{\frac{M}{1+M\frac{1-\rho}{\sqrt{e(u)}}}}^{\frac{c\sqrt{e(u)}}{1+c(1-\rho)}} \mathbb{P}\left(N_u + \frac{h\left(x(0,u)e(u)\right)}{\sqrt{x(0,u)e(u)}} > w\sqrt{x(0,u)}\right) \mathrm{d}w. \qquad (5.11)
\end{aligned}
$$

Here (5.10) can be bounded similar to (5.9). The integral (5.11) split into

$$
\int_{M}^{c\sqrt{e(u)}} + \int_{\frac{M}{1+M\frac{1-\rho}{\sqrt{e(u)}}}}^{M} - \int_{\frac{c\sqrt{e(u)}}{1+c(1-\rho)}}^{c\sqrt{e(u)}} \mathbb{P}\left(N_u + \frac{h\left(x(0,u)e(u)\right)}{\sqrt{x(0,u)e(u)}} > w\sqrt{x(0,u)}\right) \mathrm{d}w.
$$

where the last integral can be bounded as in (5.2). Further

$$
\begin{aligned}
&\int_{\frac{M}{1+M\frac{1-\rho}{\sqrt{e(u)}}}}^{M} \mathbb{P}\left(N_u + \frac{h\left(x(0,u)e(u)\right)}{\sqrt{x(0,u)e(u)}} > w\sqrt{x(0,u)}\right) \mathrm{d}w \\
&\leq \frac{1}{\sqrt{e(u)}} \frac{M^2(1-\rho)}{1+M\frac{1-\rho}{\sqrt{e(u)}}} \mathbb{P}\left(N_u + \frac{h\left(x(0,u)e(u)\right)}{\sqrt{x(0,u)e(u)}} > \frac{M\sqrt{x(0,u)}}{1+M\frac{1-\rho}{\sqrt{e(u)}}}\right) \\
&\sim \frac{M^2(1-\rho)}{\sqrt{e(u)}} \left(1-\Phi\left(\frac{M\sqrt{x(1-\rho)}}{\sqrt{\lambda \mathbb{E}\left[E^2\right]}}\right)\right).
\end{aligned}
$$

Hence the Lemma follows.

$\square$

We now provide the similar Lemmas for (5.4). We will skip the proofs, since apart from some obvious modifications they are similar to the proofs of Lemmas 5.5, 5.6 and 5.7.

Lemma 5.5. *Under Assumption 3.1 we get that for every $c > 0$ uniformly for $w < M$ that*

$$
\begin{aligned}
&\mathbb{P}\left(N_u + \frac{h\left(x(-w,u)e(u)\right)}{\sqrt{x(w,u)e(u)}} \leq -w\sqrt{x(-w,u)}\right) \\
&= \mathbb{P}\left(N_u + \frac{h\left(x(0,u)e(u)\right)}{\sqrt{x(0,u)e(u)}} \leq -w\sqrt{x(0,u)}\right) \\
&- w^2\sqrt{x(1-\rho)}\frac{(1-\rho)}{\sqrt{e(u)}} f_{0,\infty}(-w\sqrt{x(1-\rho)}) \\
&+ \frac{w(1-\rho)}{2\sqrt{e(u)}} \mathbb{E}\left[\hat{Z}_u^2\right] f'_{0,\infty}\left(-w\sqrt{x(1-\rho)}\right) + o\left(\frac{1}{\sqrt{e(u)}}\right).
\end{aligned}
$$

where $x(w,u)$ is defined by (5.5) and f_∞ is the density of a normal distribution with mean 0 and variance $\lambda\mathbb{E}\left[E^2\right]$.

Lemma 5.6. *Under Assumption 3.1 we get that for every $c>0$ uniformly for $M \le w < c\sqrt{e(u)}$ that*

$$\left|\mathbb{P}\left(N_u + \frac{h\left(x(-w,u)e(u)\right)}{\sqrt{x(-w,u)e(u)}} \le -w\sqrt{x(-w,u)}\right) - \mathbb{P}\left(N_u + \frac{h\left(x(0,u)e(u)\right)}{\sqrt{x(0,u)e(u)}} \le -w\frac{x(-w,u)}{\sqrt{x(0,u)}}\right)\right|$$
$$\le \frac{C_1}{w^2\sqrt{e(u)}} + C_2 w\sqrt{e(u)}\mathbb{P}\left(E > \frac{\epsilon_1}{2}w\sqrt{e(u)}\sqrt{(1-\rho)x(w,u)}\right) + o\left(\frac{1}{e(u)}\right)$$

where $x(w,u)$ is defined by (5.5).

Lemma 5.7. *Under Assumption 3.1 we get that*

$$\int_M^{c\sqrt{e(u)}} \mathbb{P}\left(N_u + \frac{h\left(x(0,u)e(u)\right)}{\sqrt{x(0,u)e(u)}} \le -w\frac{x(-w,u)}{\sqrt{x(0,u)}}\right)\mathrm{d}w$$
$$= \int_M^{c\sqrt{e(u)}} \mathbb{P}\left(N_u + \frac{h\left(x(0,u)e(u)\right)}{\sqrt{x(0,u)e(u)}} \le -w\sqrt{x(0,u)}\right)\mathrm{d}w + R(u,M),$$

where

$$R(u,M) \lesssim \frac{C_M}{\sqrt{e(u)}}$$

and $C_M \to 0$ as $M \to \infty$.

Lemma 5.8. *Under Assumption 3.1 we get that*

$$\int_0^{c\sqrt{xe(u)}} \mathbb{P}\left(N_u + \frac{h(xe(u))}{\sqrt{xe(u)}} \le -w\right) - \mathbb{P}\left(N_u + \frac{h(xe(u))}{\sqrt{xe(u)}} > w\right)\mathrm{d}w = o\left(\frac{1}{\sqrt{e(u)}}\right).$$

Proof. Denote with $\hat{\chi}_u$ the characteristic function of $N_u + h(xe(u))/\sqrt{xe(u)}$. From the Gil-Pelaez inversion formula we get that (c.f. [14], [21])

$$\int_0^{c\sqrt{xe(u)}} \mathbb{P}\left(N_u + \frac{h(xe(u))}{\sqrt{xe(u)}} > w\right) - \mathbb{P}\left(N_u + \frac{h(xe(u))}{\sqrt{xe(u)}} \le w\right)\mathrm{d}w$$
$$= \int_0^{c\sqrt{xe(u)}} \frac{1}{\pi}\int_0^\infty \frac{1}{s}\mathrm{Im}\left(\mathrm{e}^{-\iota ws}\hat{\chi}_u(s)\right)\mathrm{d}s + \frac{1}{\pi}\int_0^\infty \frac{1}{s}\mathrm{Im}\left(\mathrm{e}^{\iota ws}\hat{\chi}_u(s)\right)\mathrm{d}s\mathrm{d}w$$
$$= \int_0^{c\sqrt{xe(u)}} \frac{2}{\pi}\int_0^\infty \frac{\cos(ws)}{s}\mathrm{Im}(\hat{\chi}_u(s))\mathrm{d}s\mathrm{d}w$$
$$= \frac{2}{\pi}\int_0^{\epsilon\sqrt{xe(u)}} \frac{\sin(c\sqrt{x(e(u))}s)}{s^2}\mathrm{Im}(\hat{\chi}_u(s))\mathrm{d}s + \frac{2}{\pi}\int_{\epsilon\sqrt{xe(u)}}^\infty \frac{\sin(c\sqrt{x(e(u))}s)}{s^2}\mathrm{Im}(\hat{\chi}_u(s))\mathrm{d}s$$
$$= I_1(u) + I_2(u)\,,$$

where ϵ is chosen such that for $|s| < \epsilon$, $-\mathrm{Re}(\chi_E''(s)) \geq \delta_1$ for some $\delta_1 > 0$. Since there exists a $\delta > 0$ such that for all $|s| > \epsilon$, $\mathrm{Re}(1 - \chi_E(s)) \geq \delta$ (E_i is non lattice). We get for $s > \epsilon\sqrt{xe(u)}$

$$|\mathrm{Im}(\hat{\chi}_u(s))| \leq \mathrm{e}^{-\frac{s^2\sigma_u^2}{2}} \mathrm{e}^{-\delta\lambda xe(u)}.$$

and hence $I_2(u)$ goes to 0 faster than any power of $e(u)$. Denote with

$$A_1(s,u) = \lambda xe(u) \int_0^\infty \cos\left(\frac{st}{\sqrt{xe(u)}}\right) - 1 \,\mathrm{d}F_E(t),$$

$$A_2(s,u) = \lambda\sqrt{xe(u)} \int_0^\infty \sqrt{xe(u)} \sin\left(\frac{st}{\sqrt{xe(u)}}\right) - st \,\mathrm{d}F_E(t)$$

To get a bound for $I_1(u)$ we get from Lemma A.1 that we have to study the derivative of

$$\frac{1}{s^2}\mathrm{Im}(\chi_u(s)) = \frac{1}{s^2}\mathrm{e}^{-\frac{s^2\sigma_u^2}{2}} \sin\left(A_2(s,u)\right) \mathrm{e}^{A_1(s,u)}$$

which is the sum of D_1, D_2 and D_3 given by

$$D_1 = \frac{\sigma_u^2}{s}\mathrm{e}^{-\frac{s^2\sigma_u^2}{2}} \sin\left(A_2(s,u)\right) \mathrm{e}^{A_1(s,u)}$$

$$D_2 = \frac{1}{s^2}\mathrm{e}^{-\frac{s^2\sigma_u^2}{2}} \sin\left(A_2(s,u)\right) \mathrm{e}^{A_1(s,u)} \left(\lambda xe(u) \int_0^\infty \frac{-t}{\sqrt{xe(u)}} \sin\left(\frac{st}{\sqrt{xe(u)}}\right) \,\mathrm{d}F_E(t)\right)$$

$$D_3 = \frac{1}{s^2}\mathrm{e}^{-\frac{s^2\sigma_u^2}{2}} \mathrm{e}^{-\frac{s^2\sigma_u^2}{2}} \left\{ \cos\left(A_2(s,u)\right) \left(\lambda\sqrt{xe(u)} \int_0^\infty t\left(\cos\left(\frac{st}{\sqrt{xe(u)}}\right) - 1\right) \,\mathrm{d}F_E(t)\right) - \frac{2}{s}\sin\left(A_2(s,u)\right) \right\}$$

Note that

$$-\frac{A_1(s,u)}{\lambda xe(u)} = \mathrm{Re}\left(\chi_E\left(\frac{s}{\sqrt{xe(u)}}\right)\right) - 1 = -\frac{s^2}{2xe(u)}\mathrm{Re}\left(\chi_E''(\xi_{s,u})\right) \geq \delta_1 \frac{s^2}{2xe(u)}.$$

Further note that

$$A_2(s,u) = -s^2 \int_0^\infty t^2 \sin\left(\xi_{s,u,t}\right) \mathrm{d}F_E(t), \tag{5.12}$$

where $0 < \xi_{s,u,t} < \frac{st}{\sqrt{xe(u)}}$. Now for $s \leq (xe(u))^{1/4}$ we have that

$$\begin{aligned}
\left|\int_0^\infty t^2 \sin\left(\xi_{s,u,t}\right) \mathrm{d}t\right| &= \left|\int_0^{(xe(u))^{1/8}} t^2 \sin\left(\xi_{s,u,t}\right) \mathrm{d}F_E(t)\right| + \left|\int_{(xe(u))^{1/8}}^\infty t^2 \sin\left(\xi_{s,u,t}\right) \mathrm{d}F_E(t)\right| \\
&\leq \int_0^{(xe(u))^{1/8}} t^2 \sin\left((xe(u))^{-1/8}\right) \mathrm{d}F_E(t) + \int_{(xe(u))^{1/8}}^\infty t^2 \mathrm{d}F_E(t) \\
&\leq \sin\left((xe(u))^{-1/8}\right) \mathbb{E}\left[E^2\right] + \int_{(xe(u))^{1/8}}^\infty t^2 \mathrm{d}F_E(t) \to 0
\end{aligned} \tag{5.13}$$

as $u \to \infty$. Hence for every $\epsilon_1 > 0$ there exists an u_0 such that for $u > u_0$ (note that $|\sin(t)| \leq t$).

$$|D_1| \leq \begin{cases} \mathbb{E}\left[E^2\right] s \exp(-\frac{\lambda\delta_1 s^2}{2}) & s > (xe(u))^{1/4} \\ \epsilon_1 s \exp(-\frac{\lambda\delta_1 s^2}{2}) & s \leq (xe(u))^{1/4} \end{cases}$$

$$|D_2| \leq \begin{cases} s\lambda\mathbb{E}\left[E^2\right]^2 \exp(-\frac{\lambda\delta_1 s^2}{2}) & s > (xe(u))^{1/4} \\ \epsilon_1 s\lambda\mathbb{E}\left[E^2\right] \exp(-\frac{\lambda\delta_1 s^2}{2}) & s \leq (xe(u))^{1/4} \end{cases}.$$

It follows that

$$\int_0^{\epsilon\sqrt{xe(u)}} |D_1| + |D_2| \mathrm{d}s = o\left(1\right).$$

At last we have to bound $\int_0^1 |D_3|\mathrm{d}s + \int_1^{\epsilon\sqrt{xe(u)}} |D_3|\mathrm{d}s$. Since

$$\lambda\sqrt{xe(u)} \int_0^\infty t\left(\cos\left(\frac{st}{\sqrt{xe(u)}}\right) - 1\right)\ \mathrm{d}F_E(t) = -\lambda s \int_0^\infty t^2 \sin(\xi_{s,u,x})\mathrm{d}s.$$

We get with the same method as above

$$\int_1^{\epsilon\sqrt{xe(u)}} |D_3|\mathrm{d}s = o\left(1\right).$$

For $0 < s < 1$ we get with (5.12) and (5.13) that for large enough u

$$\begin{aligned}
|D_3| &\leq \frac{1}{s^2}\left|\cos\left(A_2(s,u)\right)\left(\lambda\sqrt{xe(u)}\int_0^\infty t\left(\cos\left(\frac{st}{\sqrt{xe(u)}}\right)-1\right)\ \mathrm{d}F_E(t)\right) - \frac{2}{s}\sin\left(A_2(s,u)\right)\right| \\
&= \frac{\lambda\sqrt{xe(u)}}{s^2}\left|\int_0^\infty t\left(\cos\left(\frac{st}{\sqrt{xe(u)}}\right)-1\right)\ \mathrm{d}F_E(t)\right. \\
&\quad \left. - \frac{2}{s}\int_0^\infty \sqrt{xe(u)}\sin\left(\frac{st}{\sqrt{xe(u)}}\right) - st\ \mathrm{d}F_E(t)\right| + o(1). \\
&\leq 4\frac{\lambda\sqrt{xe(u)}}{s^2}\int_{\frac{1}{s}\sqrt{x(e(u))}}^\infty t\mathrm{d}F_E(t) + \frac{2\sin(1) - 3\cos(1)}{2\sqrt{xe(u)}}\int_0^{\frac{1}{s}\sqrt{x(e(u))}} t^3\mathrm{d}F_E(t) + o(1).
\end{aligned}$$

It is left to show that $\int_0^1 \mathrm{d}s$ of the last equation is $o(1)$. From Karamata's Theorem it follows that

$$\begin{aligned}
\int_0^1 4\frac{\lambda\sqrt{xe(u)}}{s^2}\int_{\frac{1}{s}\sqrt{x(e(u))}}^\infty t\mathrm{d}F_E(t)\mathrm{d}s &\sim c\int_0^1 \frac{\lambda xe(u)}{s^3}\overline{F}_E\left(\frac{1}{s}\sqrt{x(e(u))}\right)\mathrm{d}s \\
&= c\int_{\sqrt{xe(u)}}^\infty s\overline{F}_E\left(s\right)\mathrm{d}s = o(1).
\end{aligned}$$

If $\mathbb{E}\left[E^3\right] < \infty$ then the Lemma follows. If $\mathbb{E}\left[E^3\right] = \infty$ and $\alpha \neq 3$ then

$$\int_0^1 \frac{1}{\sqrt{xe(u)}}\int_0^{\frac{1}{s}\sqrt{x(e(u))}} t^3\mathrm{d}F_E(t)\mathrm{d}s \sim c\int_0^1 \frac{\lambda xe(u)}{s^3}\overline{F}_E\left(\frac{1}{s}\sqrt{x(e(u))}\right)\mathrm{d}s = o(1).$$

If $\alpha = 3$ and $\mathbb{E}\left[E^3\right] = \infty$ then the integral is asymptotically less as when we replace t^3 with $t^{3.5}$ and the Lemma follows with the same argument. $\square$

6 The asymptotics of the Z

In what follows we will denote with $S_n = \sum_{i=1}^n Y_i$, $M_n = \max_{1\le i\le n} Y_i$, $\hat{S}_n = \sum_{i=1}^n Z_i$, and with $\hat{M}_n = \max_{1\le i\le n} Z_i$

Lemma 6.1. *Let $(Y_1, Z_1), \dots (Y_1, Z_1)$ be iid vectors with distribution $F_0(y+z)$ where $F_0 = \frac{1}{\mu}\int_0^x \overline{F}(t)\mathrm{d}t$ and F fulfill Assumption 3.1. Then*

$$\begin{aligned}
&\mathbb{P}(S_n > u, S_{n-1} \le u, \hat{S}_n > xe(u)) \sim \overline{F}_0(u + xe(u)) \\
&\quad + \frac{1}{\mu}\mathbb{E}\left[S_{n-1} + \hat{S}_{n-1} + (n-1)Y_n\right]\overline{F}(u + xe(u)) \\
&= F_0(u + xe(u)) + \frac{3(n-1)\mathbb{E}\left[X^2\right]}{\mu}\overline{F}(u + xe(u)).
\end{aligned}$$

Further for all $\epsilon > 0$ there exists a constant M such that for all n

$$\mathbb{P}(S_n > u, S_{n-1} \le u, Z > x(e(u))) - \overline{F}_0(u + xe(u)) \le M(1+\epsilon)^n\overline{F}(u).$$

Proof. Note that

$$\sum_{i=1}^n \mathbb{P}(S_n > u, S_{n-1} \le u, \hat{S}_n > xe(u), M_n = Y_i).$$

At first we consider $\{M_n = X_n\}$. We have that

$$\begin{aligned}
&\mathbb{P}(S_n > u, S_{n-1} \le u, \hat{S}_n > xe(u), M_n = Y_n) \\
&= \mathbb{P}(S_n > u, S_{n-1} \le u, \hat{S}_n > xe(u), M_n = Y_n, S_{n-1} > u/2) \\
&\quad + \mathbb{P}(S_n > u, \hat{S}_n > xe(u), M_n = Y_n, S_{n-1} \le u/2, \hat{S}_{n-1} > u/2) \\
&\quad + \mathbb{P}(S_n > u, \hat{S}_n > xe(u), M_n = Y_n, S_{n-1} \le u/2, \hat{S}_{n-1} \le u/2).
\end{aligned}$$

Since F is regularly varying case we get

$$\begin{aligned}
&\mathbb{P}(S_n > u, S_{n-1} \le u, \hat{S}_n > xe(u), M_n = Y_n, S_{n-1} > u/2) \\
&\le \mathbb{P}(Y_n > u/n)\mathbb{P}(S_{n-1} > u/2) \\
&\le K(2n)^{\alpha + d\epsilon}(1+\epsilon)^n\overline{F}_0(u)^2
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{P}(S_n > u, S_{n-1} \le u, \hat{S}_n > xe(u), M_n = Y_n, \hat{S}_{n-1} > u/2) \\
&\le \mathbb{P}(Y_n > u/n)\mathbb{P}(\hat{S}_{n-1} > xe(u)) \\
&\le K(2n)^{\alpha + d\epsilon}(1+\epsilon)^n\overline{F}_0(u)\overline{F}(xe(u)).
\end{aligned}$$

We are left with

$$
\begin{aligned}
&\mathbb{P}(S_n > u, \hat{S}_n > xe(u), M_n = Y_n, S_{n-1} \le u/2, \hat{S}_{n-1} \le xe(u)/2) \\
&= \int_0^{u/2}\int_0^{\frac{xe(u)}{2}} \overline{F}_0(u - S_{n-1} + xe(u) - \hat{S}_{n-1})\mathrm{d}S_{n-1}\mathrm{d}\hat{S}_{n-1} \\
&= \int_0^{u/2}\int_0^{\frac{xe(u)}{2}} \overline{F}_0(u + xe(u))\mathrm{d}S_{n-1}\mathrm{d}\hat{S}_{n-1} \\
&\quad + \frac{1}{\mu}\int_0^{u/2}\int_0^{\frac{xe(u)}{2}} (S_{n-1} + \hat{S}_{n-1})\overline{F}(u + xe(u) - \xi_{u,S_{n-1},\hat{S}_{n-1}})\mathrm{d}S_{n-1}\mathrm{d}\hat{S}_{n-1},
\end{aligned}
$$

where $0 < \xi_{u,S_{n-1},\hat{S}_{n-1}} < (u + xe(u))/2$ and hence there exists a constant C such that $\overline{F}(u + xe(u) - \xi_{u,S_{n-1},\hat{S}_{n-1}}) \le C\overline{F}(u + xe(u))$. It follows by dominated convergence that

$$
\begin{aligned}
&\frac{1}{\mu}\int_0^{u/2}\int_0^{\frac{xe(u)}{2}} (S_{n-1} + \hat{S}_{n-1})\overline{F}(u + xe(u) - \xi_{u,S_{n-1},\hat{S}_{n-1}})\mathrm{d}S_{n-1}\mathrm{d}\hat{S}_{n-1} \\
&\sim \frac{1}{\mu}\mathbb{E}\left[S_{n-1} + \hat{S}_{n-1}\right]\overline{F}(u + xe(u)).
\end{aligned}
$$

Note that

$$
\begin{aligned}
1 - \int_0^{u/2}\int_0^{\frac{xe(u)}{2}} \mathrm{d}S_{n-1}\mathrm{d}\hat{S}_{n-1} &\le \mathbb{P}\left(S_{n-1} > u/2\right) + \mathbb{P}\left(\hat{S}_{n-1} > \frac{xe(u)}{2}\right) \\
&\le K(1+\epsilon)^n 2^{n+\epsilon}\overline{F}_0(u)\overline{F}_0(xe(u)).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\mathbb{P}(S_n > u, S_{n-1} \le u, \hat{S}_n > xe(u), M_n = Y_n) \\
&= \mathbb{P}(S_n > u, \hat{S}_n > xe(u), M_n = Y_n) + \mathcal{O}(\overline{F}_0(u)^2) \\
&= \overline{F}_0(u + xe(u)) + \frac{1}{\mu}\mathbb{E}\left[S_{n-1} + \hat{S}_{n-1}\right]\overline{F}(u + xe(u)) + o(\overline{F}(u)).
\end{aligned} \tag{6.1}
$$

Next consider $\{M_n = Y_i\}$ where w.l.o.g we will assume that $i = n-1$.
Then we get with the same method as that leads to (6.1)

$$
\begin{aligned}
&\mathbb{P}(S_n > u, S_{n-1} < u, \hat{S}_n > xe(u), M_n = Y_{n-1}) \\
&= \mathbb{P}(Y_{n-1} > u - S_{n-2} - Y_n, \hat{S}_n > xe(u), M_n = Y_{n-1}) \\
&\quad - \mathbb{P}(Y_{n-1} > u - S_{n-2}, \hat{S}_n > xe(u), M_n = Y_{n-1}) \\
&= \frac{1}{\mu}\mathbb{E}\left[Y_n\right]\overline{F}(u + xe(u)) + o(\overline{F}(u)).
\end{aligned}
$$

□

We also need some properties of the density of Z. As in Lemma 6.1 we can get upper bounds such that with dominated convergence we can use a random n.

Lemma 6.2. *Under Assumption 3.1 we get that*

$$\left.\frac{\mathrm{d}\mathbb{P}(S_{n-1} \leq u, S_n > u, \hat{S}_n \leq x)}{\mathrm{d}x}\right|_{x=yu} \sim \frac{1}{\mu}\overline{F}(u+yu).$$

Proof. Note that

$$\mathbb{P}(\hat{S}_n > x) = \int_0^x \int_0^u \mathbb{P}(Y_n > u - S_{n-1}, Z_n \leq x - \hat{S}_{n-1})\mathrm{d}S_{n-1}\mathrm{d}\hat{S}_{n-1}$$

It follows that

$$\begin{aligned}
&\frac{\mathrm{d}\mathbb{P}(S_{n-1} \leq u, S_n > u, \hat{S}_n \leq x)}{\mathrm{d}x} \\
&= \mu^{-n} \int_{S_{n-1}<u} \int_{u-S_{n-1}}^{\infty} \int_0^x \int_0^{x-x_1} \cdots \int_0^{x-\sum_{i=1}^{n-1} x_i} \\
&\quad f(x_1+y_1)\cdots f(x_{n-1}+y_{n-1}) f\left(x - \sum_{i=1}^{n-1} x_i + y_n\right) \mathrm{d}x_1\cdots\mathrm{d}x_{n-1}\mathrm{d}y_1\cdots\mathrm{d}y_n \\
&= \mu^{-1}\mathbb{E}\left[\overline{F}\left(u+x-S_{n-1}-\hat{S}_{n-1}\right), S_{n-1} < u, \hat{S}_{n-1} < x/2\right] &(6.2)\\
&\quad + \mu^{-1}\mathbb{E}\left[\overline{F}\left(u+x-S_{n-1}-\hat{S}_{n-1}\right), S_{n-1} < u, x/2 \leq \hat{S}_{n-1} < x\right]. &(6.3)
\end{aligned}$$

If we choose $x = yu$ for some $y > 0$ then we get with dominated convergence that (6.2) $\sim \overline{F}(u+x)$ further (6.2) $\leq c(y)\overline{F}(u+x)$ for some $0 < c(y) < \infty$. To find a bound for (6.3) note that the mean over the region $Z_i > x/4n$ and $Z_j > x/4n$ can be bounded by $\overline{F}_0(x/4n)^2$ (and since $\mathbb{E}\left[X^2\right] < \infty$ we get that $\overline{F}_0(x)^2 = o(\overline{F}(x))$. By using the symmetry of the problem in the Z_i we can asymptotically bound the mean of (6.3) by

$$\begin{aligned}
&\mathbb{E}\left[\overline{F}\left(u+x-S_{n-1}-\hat{S}_{n-1}\right), S_{n-2} > u/4, S_{n-1} < u, x/2 \leq \hat{S}_{n-1} < x, \hat{S}_{n-2} \leq x/4\right] \\
&+ \mathbb{E}\left[\overline{F}\left(u+x-S_{n-1}-\hat{S}_{n-1}\right), S_{n-2} \leq u/4, S_{n-1} < u, x/2 \leq \hat{S}_{n-1} < x, \hat{S}_{n-2} \leq x/4\right].
\end{aligned}$$

If $S_{n-2} > u/4$ then one of the Y_i $i \leq n-2$ is bigger then $u/(4(n-2))$ and we can bound the corresponding mean by $\overline{F}_0(x/4)\overline{F}_0(u/(4(n-2))$.
It is left to bound the mean when $S_{n-2} \leq u/4$. At first we assume that $S_{n-1} < u/2$ then we can use the same method to bound the mean by

$$\begin{aligned}
&\mathbb{E}\left[\overline{F}\left(u+x-S_{n-1}-\hat{S}_{n-1}\right), S_{n-1} \leq u/2, x/2 \leq \hat{S}_{n-1} < x, \hat{S}_{n-2} \leq x/4\right] \\
&\leq \mathbb{E}\left[\overline{F}\left(u/2\right), Z_{n-1} > x/4\right] = \overline{F}\left(u/2\right)\overline{F}_0(x/4).
\end{aligned}$$

Finally note that

$$\mathbb{E}\left[\overline{F}\left(u+x-S_{n-1}-\hat{S}_{n-1}\right), S_{n-2}\le u/4, u/2<S_{n-1}\le u, x/2\le \hat{S}_{n-1}<x, \hat{S}_{n-2}\le x/4\right]$$

$$= \mu^{-1}\mathbb{E}\Bigg[\int_{u/2-S_{n-2}}^{u-S_{n-2}}\int_{x/2-\hat{S}_{n-2}}^{x-\hat{S}_{n-2}} f(z+y)$$

$$\times\overline{F}\left(u+x-S_{n-2}-\hat{S}_{n-2}-z-y\right)\mathrm{d}z\mathrm{d}y, S_{n-2}\le u/4, \hat{S}_{n-2}\le x/4\Bigg]$$

$$= \mu^{-1}\mathbb{E}\Bigg[\int_0^{u/2}\int_0^{x/2} f(u+x-S_{n-2}-\hat{S}_{n-2}-z-y)$$

$$\times\overline{F}(z+y)\,\mathrm{d}z\mathrm{d}y, S_{n-2}\le u/4, \hat{S}_{n-2}\le x/4\Bigg]$$

$$\lesssim \mu^{-1}f((u+x)/4)\int_0^{u/2}\overline{F}(z+y)\,\mathrm{d}z\mathrm{d}y.$$

The integral in the last equation is finite since $\mathbb{E}\left[X^2\right]<\infty$. ☐

Next we consider the derivative of the density of $\hat{S}_n$. It follows that

Lemma 6.3. *Under Assumption 3.1 we get that*

$$\left.\frac{\mathrm{d}^2\mathbb{P}(S_{n-1}\le u, S_n>u, \hat{S}_n\le x)}{\mathrm{d}x^2}\right|_{x=yu} \sim \frac{1}{\mu}f(u+yu).$$

Proof.

$$\frac{\mathrm{d}^2\mathbb{P}(S_{n-1}\le u, S_n>u, \hat{S}_n\le x)}{\mathrm{d}x^2}$$

$$= -\mu^{-1}\mathbb{E}\left[f\left(u+x-S_{n-1}-\hat{S}_{n-1}\right), S_{n-1}\le u, \hat{S}_{n-1}\le x\right]$$

$$+\mu^{-2}\mathbb{E}\left[\int_0^{u-S_{n-2}} f\left(x-\hat{S}_{n-2}+y\right)\overline{F}(u-S_{n-2}-y)\,\mathrm{d}y, S_{n-2}\le u, \hat{S}_{n-2}\le x\right]$$

$$= I_1+I_2.$$

We only give a detailed asymptotic analysis for I_2 (the asymptotic of I_1 can be found analogously). If $S_{n-1}\le u/2$ and $\hat{S}_{n-2}\le x/2$ then the mean can be asymptotically bounded by $\overline{F}(x/2)\overline{F}(u/2)=o(f(x+u))$. Next we consider the case where $S_{n-1}>u/2$ and only one $Y_i>u/(4n)$.
At first we assume that $S_{n-2}\le u/4$ and $u/2<S_{n-1}\le u$. Then

$$\mu^{-1}\mathbb{E}\left[\int_{u/2-S_{n-2}}^{u-S_{n-2}} f\left(x-\hat{S}_{n-2}+y\right)\overline{F}(u-S_{n-2}-y)\,\mathrm{d}y, S_{n-2}\le u/4, \hat{S}_{n-2}\le x\right]$$

$$= \mu^{-1}\mathbb{E}\left[\int_0^{u/2} f\left(x+u-\hat{S}_{n-2}-y-S_{n-2}\right)\overline{F}(y)\,\mathrm{d}y, S_{n-2}\le u/4, \hat{S}_{n-2}\le x\right]$$

$$\sim f(x+u)$$

where the last equation follows with dominated convergence. If $Y_{n-1} \le u/(4n)$ then by symmetry it is enough to consider $Y_{n-2} > u/4$. Hence we get

$$\mu^{-2}\mathbb{E}\Bigg[\int_0^{u/4-S_{n-3}}\int_0^{x-\hat{S}_{n-3}}\int_{u/2-S_{n-3}-y}^{u-S_{n-3}-y} f(x_{n-2}+y_{n-2})f\left(x-\hat{S}_{n-3}-z_{n-2}+y\right)$$
$$\times\, \overline{F}\left(u-S_{n-3}-y_{n-2}-y\right)\mathrm{d}y_{n-2}\mathrm{d}z_{n-2}\mathrm{d}y, S_{n-3}\le u/4, \hat{S}_{n-3}\le x\Bigg]$$
$$= \mu^{-2}\mathbb{E}\Bigg[\int_0^{u/4-S_{n-3}}\int_0^{x-\hat{S}_{n-3}}\int_0^{u/2} f(x_{n-2}+u-S_{n-3}-y_{n-2}-y)$$
$$\times\, f\left(x-\hat{S}_{n-3}-z_{n-2}+y\right)\overline{F}\left(y_{n-2}\right)\mathrm{d}y_{n-2}\mathrm{d}z_{n-2}\mathrm{d}y, S_{n-3}\le u/4, \hat{S}_{n-3}\le x\Bigg].$$

For $\hat{S}_{n-2} \le x/2$ the above mean is $\mathcal{O}(\overline{F}(x/2)\overline{F}(u/4)) = o(f(u+x))$. If $x/2 < \hat{S}_{n-2} < x$ and $Z_i \le x/4n$ for all but one $i \ne n-2$ the above mean is $\mathcal{O}(\overline{F}(x/4x)\overline{F}(u/4))$. If more then two $Z_i > x/4n$ $i \ne n-2$ the above mean is $\mathcal{O}(\overline{F}_0(x/4n)^2\overline{F}(u/4))$. If $Z_{n-2} > x/4n$ and another $Z_i > x/4n$ then the mean is $\mathcal{O}(\overline{F}_0(x/4n)\overline{F}(3u/4))$. Finally if all $Z_i \le x/4n$. then the above integral is asymptotically the same as $f(u+x)$. Similar we can show that when at least two $Y_i > u/4n$ the integral is asymptotically negligibly and hence $I_2 \sim \mu^{-1}(n-1)f(u+x)$. With the same method we get that $I_1 \sim -\mu^{-1}nf(u+x)$ and hence the Lemma follows.
Again we get that for $\hat{S}_{n-1} \le x/2$ that $I_1 \sim f(u+x)$ when $x = yu$.

□

A Some auxiliary lemmas

Lemma A.1. *Assume that for a function $g_u(x)$ such that $\sup_{x,u}|g_u(x)| < \infty$, there exists a function $h(x)$ with $|g_u'(x)| \le h(x)$ for all $u > 0$. Then for every function $a(u)$ we have as $u \to \infty$*

$$\left|\int_0^{a(u)} \sin(ux)g_u(x)\mathrm{d}x\right| \le \frac{1}{u}\int_0^{a(u)} h(x)\mathrm{d}x + o(1).$$

Proof. The Lemma follows by partial integration:

$$\int_0^{a(u)} \sin(ux)g_u(x)\mathrm{d}x = \frac{1}{u}g_u(0) - \frac{\cos(ua(u))}{u}g_u(a(u)) + \frac{1}{u}\int_0^{a(u)}\cos(ux)g_u'(x)\mathrm{d}x.$$

□

Lemma A.2. *Assume that E is non lattice and that $\mathbb{E}\left[E^2\right] < \infty$ and $h(z) = \sum_{i=1}^{N(z)} E_i - \lambda z\mathbb{E}\left[E\right]$ and N_u a normal random variable with mean zero and variance $\sigma^2 \sim e(u)^{-k}$ for some $c > 0$, $k > 0$. Then the random variable $N_u + h(xe(u))/\sqrt{xe(u)}$ has a differentiable density f_u. Further, if a, b are arbitrary but fixed, it holds uniformly for w and $0 < a < x < b < \infty$ that*

$$\lim_{u\to\infty} f_u(w) = \frac{\exp\left(-\frac{w^2}{\lambda\mathbb{E}[E^2]}\right)}{\sqrt{2\pi\lambda\mathbb{E}\left[E^2\right]}}, \quad \lim_{u\to\infty} f_u'(w) = \frac{-2w\exp\left(-\frac{w^2}{\lambda\mathbb{E}[E^2]}\right)}{\sqrt{2\pi(\lambda\mathbb{E}\left[E^2\right])^3}}.$$

If further $k \geq 4$ then

$$\left|\mathbb{P}\left(N_u + \frac{h(xe(u))}{\sqrt{xe(u)}} > w\right) - \mathbb{P}\left(\frac{h(xe(u))}{\sqrt{xe(u)}} > w\right)\right| = o\left(\frac{1}{e(u)}\right).$$

Proof. Denote with $\chi_E(s)$ the characteristic function of E and with $\sigma^2 = \lambda\mathbb{E}\left[E^2\right]$. Note that the Fourier transform of $f_u'(w) - f_{N(0,\sigma^2)}'(w)$ is

$$is\left(\mathrm{e}^{-\frac{\sigma_u^2 s^2}{2}}\mathrm{e}^{\lambda xe(u)\left(\chi_E\left(s/\sqrt{xe(u)}\right)-1\right)-i\sqrt{xe(u)}\lambda\mathbb{E}[E]s} - \mathrm{e}^{-\frac{\sigma^2 s^2}{2}}\right)$$

and hence

$$\begin{aligned}&|f_u'(w) - f_{N(0,\sigma^2)}'(w)|\\ &\leq \int_{-\infty}^{\infty}|s|\left|\mathrm{e}^{-\frac{\sigma_u^2 s^2}{2}}\mathrm{e}^{\lambda xe(u)\left(\chi_E\left(s/\sqrt{xe(u)}\right)-1\right)-i\sqrt{xe(u)}\lambda\mathbb{E}[E]s} - \mathrm{e}^{-\frac{\sigma^2 s^2}{2}}\right|\mathrm{d}s.\end{aligned}$$

Choose an $\epsilon > 0$ such that for $|x| \leq \epsilon$, $\mathrm{Re}(\chi_E''(x))$ is bounded away from 0. Since there exists a $\delta > 0$ such that for all $|s| > \epsilon$, $\mathrm{Re}(1 - \chi_E(s)) \geq \delta$ (E is non lattice).

$$\begin{aligned}&\int_{\epsilon\sqrt{xe(u)}}^{\infty}|s|\left|\mathrm{e}^{-\frac{\sigma_u^2 s^2}{2}}\mathrm{e}^{\lambda xe(u)\left(\chi_E\left(s/\sqrt{xe(u)}\right)-1\right)-i\sqrt{xe(u)}\lambda\mathbb{E}[E]s} - \mathrm{e}^{-\frac{\sigma^2 s^2}{2}}\right|\mathrm{d}s\\ &\leq \mathrm{e}^{-\delta\lambda xe(u)}\int_{\epsilon\sqrt{xe(u)}}^{\infty} s\mathrm{e}^{-\frac{\sigma_u^2 s^2}{2}}\mathrm{d}s + \int_{\epsilon\sqrt{xe(u)}}^{\infty} s\mathrm{e}^{-s^2}\mathrm{d}s\\ &\leq \frac{1}{\sigma_u^2}\mathrm{e}^{-\delta\lambda xe(u)}\int_0^{\infty} s\mathrm{e}^{-\frac{s^2}{2}}\mathrm{d}s + \int_{\epsilon\sqrt{xe(u)}}^{\infty} s\mathrm{e}^{-s^2}\mathrm{d}s \to 0\end{aligned}$$

as $u \to \infty$. With the same arguments

$$\int_{-\infty}^{-\epsilon\sqrt{xe(u)}}|s|\left|\mathrm{e}^{-\frac{\sigma_u^2 s^2}{2}}\mathrm{e}^{\lambda xe(u)\left(\chi_E\left(s/\sqrt{xe(u)}\right)-1\right)-i\sqrt{xe(u)}\lambda\mathbb{E}[E]s} - \mathrm{e}^{-\frac{\sigma^2 s^2}{2}}\right|\mathrm{d}s \to 0.$$

Further for a $\xi_{u,s}$ bounded away from 0 and $\xi_{u,s} \to \mathbb{E}\left[E^2\right]$ for fixed s as $u \to \infty$

$$\begin{aligned}&\int_{-\epsilon\sqrt{xe(u)}}^{\epsilon\sqrt{xe(u)}}|s|\left|\mathrm{e}^{-\frac{\sigma_u^2 s^2}{2}}\mathrm{e}^{\lambda xe(u)\left(\chi_E\left(s/\sqrt{xe(u)}\right)-1\right)-i\sqrt{xe(u)}\lambda\mathbb{E}[E]s} - \mathrm{e}^{-\frac{\sigma^2 s^2}{2}}\right|\mathrm{d}s\\ &= \int_{-\epsilon\sqrt{xe(u)}}^{\epsilon\sqrt{xe(u)}}|s|\left|\mathrm{e}^{-\frac{\sigma_u^2 s^2}{2}}\mathrm{e}^{-\lambda\xi_{s,u}s^2/2} - \mathrm{e}^{-\frac{\sigma^2 s^2}{2}}\right|\mathrm{d}s.\end{aligned}$$

By dominated convergence we get that the last integral tends to 0 as $u \to \infty$.
Since the estimate of $|f_u(w) - f_{N(0,\sigma^2)}(w)|$ works with exactly the same arguments we leave it to the reader.
Denote with χ_u is characteristic function of $h(xe(u))/\sqrt{xe(u)}$. Since we can find an m

such that $f_u(w) \leq m$ for all w and u we get by Lemma XVI.4 2 of [13] that

$$\left|\mathbb{P}\left(N_u + \frac{h(xe(u))}{\sqrt{xe(u)}} > w\right) - \mathbb{P}\left(\frac{h(xe(u))}{\sqrt{xe(u)}} > w\right)\right|$$
$$\leq \frac{1}{\pi}\int_{-T}^{T}\left|\frac{\left(1-\mathrm{e}^{-\frac{s^2\sigma_u^2}{2}}\right)\chi_u(s)}{s}\right|\mathrm{d}s + \frac{24m}{\pi T} \leq \frac{1}{\pi}\int_{-T}^{T}\left|\frac{s\sigma_u^2}{2}\right|\mathrm{d}s + \frac{24m}{\pi T} = \frac{\sigma_u^2 T^2}{2\pi} + \frac{24m}{\pi T}.$$

For $T = e(u)^{1+\epsilon}$ for some $0 < \epsilon < 1/2$ and $\sigma_u^2 \leq e(u)^{-4}$ we get that

$$\left|\mathbb{P}\left(N_u + \frac{h(xe(u))}{\sqrt{xe(u)}} > w\right) - \mathbb{P}\left(\frac{h(xe(u))}{\sqrt{xe(u)}} > w\right)\right| \leq \frac{e(u)^{-2+2\epsilon}}{2\pi} + \frac{24m}{\pi e(u)^{1+\epsilon}}.$$

□

Lemma A.3. *Under Assumption 3.1, let* $h(z) = \sum_{i=1}^{N(z)} E_i - \lambda z\mathbb{E}[E_i]$ *and let* N_u *be a normal random variable with mean zero and variance* $\sigma^2 \sim e(u)^{-k}$ *for some* $c > 0$, $k > 0$. *Then the random variable* $N_u + h(xe(u))/\sqrt{xe(u)}$ *has a differentiable density* f_u. *Further, if* a, b *are arbitrary but fixed it holds uniformly for* w *and* $0 < a < x < b < \infty$ *that*

$$w^3 f_u'(w) \text{ and } w^2 f_u(w)$$

are bounded for $w > w_0 > 0$ *and all* $u > u_0$ *where* u_0 *is choosen such that* $xe(u) > 1$.

Proof. Denote with $\hat{F}_E(s) = \mathbb{E}\left[\mathrm{e}^{-sE_i}\right]$ and with

$$A(s,u) = \lambda\sqrt{xe(u)}\hat{F}_E'\left(s/\sqrt{xe(u)}\right) + \sqrt{xe(u)}\lambda\mathbb{E}[E].$$

Note that the (bilateral) Laplace transform of transform of $w^3 f_u'(w)$ is given by

$$\begin{aligned}
\hat{L}_{w^3 f_u'}(s) &= \frac{\mathrm{d}}{\mathrm{d}s^3}\left(s\mathrm{e}^{\frac{\sigma_u^2 s^2}{2}}\mathrm{e}^{\lambda xe(u)\left(\hat{F}_E\left(s/\sqrt{xe(u)}\right)-1\right)+\sqrt{xe(u)}\lambda\mathbb{E}[E]s}\right)\\
&= \mathrm{e}^{\frac{\sigma_u^2 s^2}{2}}\mathrm{e}^{\lambda xe(u)\left(\hat{F}_E\left(s/\sqrt{xe(u)}\right)-1\right)+\sqrt{xe(u)}\lambda\mathbb{E}[E]s}\\
&\quad\times\Bigg\{\left(sA(s,u)+\sigma_u^2 s^2\right)^2\left(1+sA(s,u)+\sigma_u^2 s^2\right)\\
&\quad+\left(sA(s,u)+\sigma_u^2 s^2\right)\left(A(s,u)+2\sigma_u^2 s+\lambda s\hat{F}_E''\left(s/\sqrt{xe(u)}\right)\right)\\
&\quad+\left(1+2sA(s,u)+2\sigma_u^2 s^2\right)\left(A(s,u)+2\sigma_u^2 s+\lambda s\hat{F}_E''\left(s/\sqrt{xe(u)}\right)\right)\\
&\quad+\left(\lambda\hat{F}_E''\left(s/\sqrt{xe(u)}\right)+2\sigma_u^2+\lambda\frac{s}{\sqrt{xe(u)}}\hat{F}_E'''\left(s/\sqrt{xe(u)}\right)\right)\Bigg\}.
\end{aligned}$$

Note that for every $w > w_0$ and $0 < \epsilon < 1$

$$w^3 f_u'(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{w(\epsilon/w+\iota s)}\hat{L}_{w^3 f_u'}(\epsilon/w+\iota s)\mathrm{d}s.$$

Since

$$A(s,u) = \lambda\sqrt{xe(u)}\hat{F}'_E\left(s/\sqrt{xe(u)}\right) + \sqrt{xe(u)}\lambda\mathbb{E}\left[E\right] = \lambda s\hat{F}''_E\left(\xi_{s,u}\right)$$

$|\hat{F}''_E(s)| \leq \mathbb{E}\left[E^2\right]$ and $s\hat{F}'''_E(s)$ is bounded (see Lemma A.4 below) for $|s| < 1$, we get that for $|s| < 1$ the term in the curly brackets can be bounded by a polynomial in $|s|$. Hence the Lemma follows analogously to the proof of A.2. □

Lemma A.4. *Under Assumption 3.1 $s\hat{F}'''_E(s)$ is uniformly bounded for $s \to 0$*

Proof. Note that

$$E \stackrel{d}{=} X + \sum_{i=1}^{N(X)} E_i$$

and hence

$$\hat{F}_E(s) = \mathbb{E}\left[\mathrm{e}^{-sX+\lambda X(\hat{F}_E(s)-1)}\right] = \hat{F}(s - \lambda(\hat{F}_E(s)-1))$$

Since for $\mathrm{Re}(s) > 0$, $|\hat{F}_E(s)| < 1$ and hence $\mathrm{Re}(s - \lambda(\hat{F}_E(s)-1)) > 0$ hence the above formula is valid for all $\mathrm{Re}(s) > 0$. Hence both sides are infinitely often differentiable for all $\mathrm{Re}(s) > 0$ and we have

$$\begin{aligned}
\hat{F}'_E(s) &= \frac{\hat{F}'\left(s - \lambda(\hat{F}_E(s)-1)\right)}{1 + \lambda\hat{F}'\left(s - \lambda(\hat{F}_E(s)-1)\right)},\\
\hat{F}''_E(s) &= \frac{\hat{F}''\left(s - \lambda(\hat{F}_E(s)-1)\right)\left(1 - \lambda\hat{F}'_E(s)\right)^2}{1 + \lambda\hat{F}'\left(s - \lambda(\hat{F}_E(s)-1)\right)},\\
\hat{F}'''_E(s) &= \frac{\hat{F}'''\left(s - \lambda(\hat{F}_E(s)-1)\right)\left(1 - \lambda\hat{F}'_E(s)\right)^3}{1 + \lambda\hat{F}'\left(s - \lambda(\hat{F}_E(s)-1)\right)}\\
&\quad - \frac{2\lambda\hat{F}''_E(s)F''\left(s - \lambda(\hat{F}_E(s)-1)\right)\left(1 - \lambda\hat{F}'_E(s)\right)}{1 + \lambda\hat{F}'\left(s - \lambda(\hat{F}_E(s)-1)\right)}\\
&\quad - \frac{\lambda\hat{F}''_E(s)F''\left(s - \lambda(\hat{F}_E(s)-1)\right)\left(1 - \lambda\hat{F}'_E(s)\right)}{1 + \lambda\hat{F}'\left(s - \lambda(\hat{F}_E(s)-1)\right)}.
\end{aligned}$$

Since $\lambda\mathbb{E}\left[X\right] < 1$ we have that

$$\sup_{\mathrm{Re}(s)\geq 0}\left|\lambda\hat{F}'\left(s - \lambda(\hat{F}_E(s)-1)\right)\right| < 1$$

and hence $\hat{F}'_E(s)$ is bounded for all $\mathrm{Re}(s) > 0$ and since $\mathbb{E}\left[X^2\right] < \infty$ also $\hat{F}''_E(s)$ is bounded. Finally we get that $s\hat{F}'''_E(s)$ is bounded Since $s\hat{F}'''(s)$ is bounded and

$$s - \lambda(\overline{F}_E(s)-1) = s - \lambda(\hat{F}_E(s)-1) = s(1 - \lambda\hat{F}'_E(s)) + \frac{s^2}{2}\hat{F}''_E(\xi_s) = \mathcal{O}(s).$$

□

Lemma A.5. *Let E_i be iid with $\mathbb{E}[E] < \infty$ and $N(t)$ a Poisson process with intensity λ independent of the E_i. Then there exists constants C_1, C_2 and $\delta > 0$ such that uniformly for $x > \epsilon t$*

$$\mathbb{P}\left(\left|\sum_{1=1}^{N(t)} E_i - \lambda t \mathbb{E}[E]\right| > x\right) \leq C_1 t \mathbb{P}(E > x) + e^{-\delta\left(x - \frac{\epsilon}{2}t\right)}.$$

Proof. In [16] it is proved that

$$\mathbb{P}\left(\sum_{1=1}^{N(t)} E_i - \lambda t \mathbb{E}[E] > x\right) \leq C_1 t \mathbb{P}(E > x)$$

uniformly for $x > \epsilon t$. We can find a $\delta > 0$ such that for all $t > 0$

$$\mathbb{E}\left[\exp\left(-\delta\left(\sum_{1=1}^{N(t)} E_i - t\left(\lambda \mathbb{E}[E] + \frac{\epsilon}{2}\right)\right)\right)\right] \leq 1.$$

The Lemma follows by the Chernoff bound. □

We often used the following Lemma without further mentioning. Since we don't have a reference by hand we give for completeness a proof .

Lemma A.6. *Let $L(x)$ be slowly varying and*

$$\int_0^\infty \frac{1}{x} L(x) \mathrm{d}x < \infty,$$

then $\lim_{x\to\infty} L(x) = 0$

Proof. Assume that the Lemma is not true, i.e. there exists a series of points x_n with $x_n \to \infty$ and $L(x_n) > \delta$. W.l.o.g. assume that

$$\inf_{1 \leq t \leq 2} \frac{L(t x_n)}{L(x_n)} > 1/2.$$

Then

$$\int_{x_n}^{2x_n} \frac{1}{x} L(x) \mathrm{d}x \geq \frac{\delta}{2} \int_{x_n}^{2x_n} \frac{1}{x} \mathrm{d}x = \frac{\delta \log(2)}{2},$$

which contradicts the conditions of the Lemma.

□

Acknowledgment We would like to thank Jens Ledet Jensen for pointing out the proof of Lemma A.1.

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